

# Lipschitz functions on the infinite-dimensional torus

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## Abstract

We discuss the spectrum phenomenon for Lipschitz functions on the infinite-dimensional torus. Suppose that  $f$  is a measurable, real-valued, Lipschitz function on the torus  $\mathbb{T}^\infty$ . We prove that there exists a number  $a \in \mathbb{R}$  with the following property: For any  $\varepsilon > 0$  there exists a parallel, infinite-dimensional subtorus  $M \subseteq \mathbb{T}^\infty$  such that the restriction of the function  $f - a$  to the subtorus  $M$  has an  $L^\infty(M)$ -norm of at most  $\varepsilon$ .

## 1 Background and Results

One of the most remarkable phenomena in high dimensions is the emergence of a *spectrum* for uniformly continuous functions. It was shown by Milman in his proof of Dvoretzky's theorem [6] that given any 1-Lipschitz function  $f$  on the high-dimensional sphere  $S^n$ , one may find a section of  $S^n$  by a linear subspace of large dimension, on which  $f$  is nearly a constant function. The value of this constant may be thought of, approximately, as an element in a spectrum associated with  $f$ . An analogous effect in discrete mathematics is Ramsey's theorem [1], according to which any coloring of a large complete graph by a fixed number of colors contains a large induced subgraph which is monochromatic.

There have been several attempts to formulate infinite-dimensional analogs of the Ramsey-Dvoretzky-Milman phenomenon. Let  $X$  be an infinite-dimensional Banach space whose unit sphere is denoted by  $S(X)$ . For a function  $f : S(X) \rightarrow \mathbb{R}$  one defines its infinite-dimensional spectrum  $\sigma(f)$  as the collection of all values  $a \in \mathbb{R}$  with the following property: For any  $\varepsilon > 0$ , there exists an infinite-dimensional subspace  $Y \subseteq X$  such that

$$|f(v) - a| < \varepsilon \quad \text{for all } v \in S(Y),$$

where  $S(Y)$  is the unit sphere in the subspace  $Y$ . A question that was open for many years was whether the infinite-dimensional spectrum of any Lipschitz function is non-empty. Unfortunately, even when  $X$  is a Hilbert space, the answer is decisively negative as was proven

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by Odell and Schlumprecht [7, 8]. On the positive side, Gowers [3] proved that any Lipschitz function on the Banach space  $c_0$  admits a non-empty infinite-dimensional spectrum. The space  $c_0$  is essentially the only separable Banach space for which the answer is positive, as was proven in [7, 8].

Here we investigate the question of existence of an infinite-dimensional spectrum in a different situation, that of the infinite-dimensional torus, or of infinite-dimensional product spaces in general. Lipschitz functions on a finite-dimensional torus were analyzed using probabilistic tools by Faifman, Klartag and Milman [2]. In this paper we will exploit the fact that the infinite-dimensional torus admits a product probability measure, which allows one to use probabilistic arguments akin to the finite-dimensional case.

Let us introduce some terminology and notation and recall a few basic facts that are used throughout the paper. The infinite-dimensional torus is typically denoted by  $\mathbb{T}^{\mathbb{N}}$  or by  $\mathbb{T}^{\infty}$ . An element  $x \in \mathbb{T}^{\infty}$  is a sequence  $x = (x_i)_{i=1,2,\dots}$  with  $x_i \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$  for all  $i$ . Write  $\sigma$  for the uniform probability measure on  $\mathbb{T}^{\infty}$ , which is a complete product measure, invariant under translations. When we say that a random point  $X$  is distributed uniformly on  $\mathbb{T}^{\infty}$  or when we say that a function  $f$  on  $\mathbb{T}^{\infty}$  is measurable, we always refer to the probability measure  $\sigma$ . For  $x, y \in \mathbb{T}^{\infty}$  consider the Euclidean metric

$$\text{dist}(x, y) = \sqrt{\sum_{i=1}^{\infty} \text{dist}^2(x_i, y_i)}$$

where  $x = (x_i)_{i \geq 1}, y = (y_i)_{i \geq 1}$  and where  $\text{dist}(x_i, y_i)$  is the distance between  $x_i$  and  $y_i$  in the circle  $\mathbb{R}/\mathbb{Z}$ . It may happen that  $d(x, y) = +\infty$  for some  $x, y \in \mathbb{T}^{\infty}$ . In fact, the torus  $\mathbb{T}^{\infty}$  is split into infinitely-many connected components with respect to the metric  $\text{dist}$ , all of measure zero. It is explained in Gromov [4, Section 3.1] that for any measurable subsets  $A, B \subseteq \mathbb{T}^{\infty}$  with  $\sigma(A) \cdot \sigma(B) > 0$ ,

$$\inf_{x \in A, y \in B} \text{dist}(x, y) \leq C(\sigma(A), \sigma(B)) < \infty$$

for a certain explicit function  $C : (0, 1] \times (0, 1] \rightarrow [0, \infty)$ . A subset  $M \subseteq \mathbb{T}^{\infty}$  is a *parallel infinite-dimensional subtorus* if there exist an infinite subset  $A \subseteq \mathbb{N}$  and values  $b : \mathbb{N} \setminus A \rightarrow \mathbb{T}$  such that

$$M = \{(x_i)_{i \geq 1} \in \mathbb{T}^{\infty} ; x_i = b_i \text{ for all } i \in \mathbb{N} \setminus A\}.$$

Note that the uniform probability measure on the infinite-dimensional subtorus  $M$  is well-defined, thus one may speak of the space  $L^{\infty}(M)$ . Our main result is the following:

**Theorem 1.** *For any measurable function  $f : \mathbb{T}^{\infty} \rightarrow \mathbb{R}$  that is Lipschitz with respect to the Euclidean metric  $\text{dist}$ , there exists  $a \in \mathbb{R}$  with the following property: For any  $\varepsilon > 0$  there exists a parallel infinite-dimensional subtorus  $M \subseteq \mathbb{T}^{\infty}$  such that  $\|f - a\|_{L^{\infty}(M)} < \varepsilon$ .*

Theorem 1 thus implies that any measurable, Lipschitz function on  $\mathbb{T}^{\infty}$  has a non-empty spectrum in an appropriate sense. In order to have in mind some concrete examples of measurable functions on  $\mathbb{T}^{\infty}$ , we mention the function

$$f(x) = \sum_{i=1}^{\infty} a_i \cos(2\pi x_i) \quad (x \in \mathbb{T}^{\infty}) \quad (1)$$

where  $\cos(2\pi x_i)$  is clearly well-defined for  $x_i \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ . By Kolmogorov's three-series theorem (see, e.g., Kahane [5, Section 3]), the series in (1) converges almost everywhere if and only if  $\sum a_i^2 < \infty$ . Assuming that indeed  $\sum a_i^2 < \infty$ , the function  $f$  is a well-defined<sup>1</sup>, measurable function on  $\mathbb{T}^\infty$  which is in fact Lipschitz with respect to the Euclidean metric  $\text{dist}$ .

We proceed to discuss the necessity of the assumptions of Theorem 1. The condition of measurability is essential: Indeed, fixing a representative  $x^C$  for each dist-connected component  $C$  of  $\mathbb{T}^\infty$ , and letting  $f(x) = \inf_C \text{dist}(x, x^C)$ , we get a Lipschitz function, the restriction of which to every parallel, infinite-dimensional subtorus has arbitrarily large values. An example of a measurable function which is non-Lipschitz and has an empty spectrum may be constructed as follows: It is well-known that there exists a Borel subset  $B \subset \mathbb{R}$  such that both  $B$  and  $\mathbb{R} \setminus B$  intersect any non-empty interval in a set of a positive Lebesgue measure. Consider the set

$$A = \left\{ (x_1, x_2, \dots) \in \mathbb{T}^\infty ; \sum_{i=1}^{\infty} \frac{\cos(2\pi x_i)}{i^2} \in B \right\}.$$

Then the indicator function  $f = 1_A$  is a measurable function which has no spectrum.

In general, a measurable, dist-Lipschitz function need not be continuous with respect to the usual product topology on  $\mathbb{T}^\infty$ . The function in (1) is continuous with respect to the product topology only under the stronger requirement that  $\sum |a_i| < \infty$ . For a function  $f : \mathbb{T}^\infty \rightarrow \mathbb{R}$  that is continuous in the product topology, its image coincides with its spectrum. This is because every element of the basis of the topology contains a parallel infinite-dimensional subtorus of the form  $M = \{(x_i)_{i \geq 1} \in \mathbb{T}^\infty ; x_i = b_i \ \forall i < N\}$ .

In addition to the Euclidean metric  $\text{dist}$ , one defines for  $1 \leq p \leq \infty$  and  $x, y \in \mathbb{T}^\infty$  the distance  $\text{dist}_p$  by

$$\text{dist}_p(x, y) = \left( \sum_{i=1}^{\infty} \text{dist}^p(x_i, y_i) \right)^{1/p}, \quad (2)$$

where the case  $p = \infty$  is defined by  $\text{dist}_\infty(x, y) = \sup_{i \geq 1} \text{dist}(x_i, y_i)$ . Theorem 1 is the case  $p = 2$  of the following:

**Theorem 2.** *For any  $1 < p \leq \infty$  and a measurable function  $f : \mathbb{T}^\infty \rightarrow \mathbb{R}$  which is Lipschitz with respect to the metric  $\text{dist}_p$ , there exists  $a \in \mathbb{R}$  with the following property: For any  $\varepsilon > 0$  there exists a parallel infinite-dimensional subtorus  $M \subseteq \mathbb{T}^\infty$  such that  $\|f - a\|_{L^\infty(M)} < \varepsilon$ .*

It is only the product structure of  $\mathbb{T}^\infty$  that plays a fundamental role in the proof of Theorem 2 given below. For instance, one may replace the infinite product of circles  $\mathbb{T}^\infty$  by the infinite-dimensional cube  $[0, 1]^\infty$ , or more generally, by an infinite product of the form

$$X = X_1 \times X_2 \times \dots$$

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<sup>1</sup>Formally,  $f$  is well-defined only almost everywhere with respect to  $\sigma$ . For completeness, let us agree that  $f$  attains the value zero at the few points  $x$  for which the series in (1) diverges.

where  $X_1, X_2, \dots$  are connected Riemannian manifolds with boundary, all of volume one, that have a “uniformly bounded geometry”. By the last phrase we mean that the dimensions, diameters and sectional curvatures of the  $X_i$ ’s should all be uniformly bounded. The distance function  $\text{dist}_p$  on  $X$  is still given by (2). For concreteness, we provide the statement and proof only for the toric case. We believe that the adaptation of our proof of Theorem 2 to the cube  $[0, 1]^\infty$  or to the case of a more general product space is rather straightforward.

We are not sure whether the conclusion of Theorem 2 holds true also for  $p = 1$ . It could be interesting to investigate whether for  $p = \infty$ , the essential supremum in the conclusion of Theorem 2 may be replaced by a supremum. Let us also comment that the full axiom of choice is not used in the proof of Theorem 2, and that the axiom of dependent choice suffices for our argument.

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## 2 Proofs

Consider the  $n$ -dimensional torus  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ . The coordinate vector fields  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  are well-defined on the torus  $\mathbb{T}^n$ . The metric  $\text{dist}_p$  on the finite-dimensional torus  $\mathbb{T}^n$  is defined via a formula analogous to (2) in which the sum runs only up to  $n$ . For a function  $f : \mathbb{T}^n \rightarrow \mathbb{R}$  we define its oscillation via

$$\text{Osc}(f; \mathbb{T}^n) = \sup_{\mathbb{T}^n} f - \inf_{\mathbb{T}^n} f = \sup_{x, y \in \mathbb{T}^n} |f(x) - f(y)|.$$

According to the Rademacher theorem from real analysis, any function on  $\mathbb{T}^n$  which is Lipschitz with respect to  $\text{dist}_p$ , for some  $1 \leq p \leq \infty$ , is differentiable almost-everywhere. Let  $\omega_{n,p}$  denote the  $n$ -dimensional volume of the  $\ell_p$ -ball  $B_p^n = \{x \in \mathbb{R}^n ; \sum_{i=1}^n |x_i|^p \leq 1\}$ . In this note, all integrals on tori and subtori are carried out with respect to the uniform probability measure on the torus. We will need the following variant of Morrey’s inequality:

**Lemma 3.** *Let  $n \geq 1, p \in (1, \infty], 0 < \varepsilon < 1/2$  and let  $f : \mathbb{T}^n \rightarrow \mathbb{R}$  be 1-Lipschitz with respect to the metric  $\text{dist}_p$ . Denote  $q = p/(p-1)$ , with  $q = 1$  in case  $p = \infty$ . Assume that*

$$\int_{\mathbb{T}^n} \sum_{i=0}^{n-1} \frac{2^{i^2+qi}}{\omega_{i,p} \varepsilon^{i+q}} \cdot \left| \frac{\partial f}{\partial x_{i+1}} \right|^q \leq 1. \quad (3)$$

*Then  $\text{Osc}(f; \mathbb{T}^n) < 8\varepsilon$ .*

*Proof.* Let  $\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  be the quotient map. For a point  $x \in \mathbb{T}^n$  we denote

$$\ell_i = \pi(\mathbb{R}e_i), \quad E_i = \pi(\text{Sp}(e_1, \dots, e_i)),$$

where  $e_1, \dots, e_n$  are the standard unit vectors in  $\mathbb{R}^n$  and where  $\text{Sp}(e_1, \dots, e_i)$  is the subspace spanned by  $e_1, \dots, e_i$ . We also denote  $x + A = \{x + y ; y \in A\}$  for a subset  $A \subseteq \mathbb{T}^n$  and a point  $x \in \mathbb{T}^n$ . Thus,  $x + \ell_i \subseteq \mathbb{T}^n$  is a one-dimensional torus in  $\mathbb{T}^n$  passing through  $x$  in

the direction of  $\partial/\partial x_i$ . The subtorus  $x + E_i \subseteq \mathbb{T}^n$  is  $i$ -dimensional, and the vector fields  $\partial/\partial x_1, \dots, \partial/\partial x_i$  are tangent to  $x + E_i$  at the point  $x$ .

Fix a point  $P \in \mathbb{T}^n$ . For a decreasing index  $i = n, \dots, 0$ , we recursively define the random points  $P_i, P'_i \in \mathbb{T}^n$  via the following rules:

- (i)  $P_n = P$ .
- (ii) The point  $P'_i$  is distributed uniformly in the  $i$ -dimensional ball  $B_p^i(P_{i+1}, \frac{\varepsilon}{2^i})$ , where  $B_p^i(P_{i+1}, \frac{\varepsilon}{2^i})$  is the  $\text{dist}_p$ -ball in the subtorus  $P_{i+1} + E_i$  centered at  $P_{i+1}$  of radius  $\frac{\varepsilon}{2^i}$ .
- (iii) The point  $P_i$  is distributed uniformly in the 1-dimensional subtorus  $P'_i + \ell_{i+1}$ .

Note that our recursive definition has a decreasing index, thus we first define  $P_n$ , then  $P'_{n-1}$ , then  $P_{n-1}$ , etc. Since  $f$  is Lipschitz, for  $i = 0, \dots, n-1$ ,

$$\mathbb{E}|f(P'_i) - f(P_i)| \leq \mathbb{E} \int_{P'_i + \ell_{i+1}} \left| \frac{\partial f}{\partial x_{i+1}} \right| = \mathbb{E} \left| \frac{\partial f}{\partial x_{i+1}}(P_i) \right|. \quad (4)$$

By an inductive argument, we see that the last  $n-i$  coordinates of the random point  $P_i$  are independent random variables that are distributed uniformly over the circle  $\mathbb{T}$ . Let  $A_{i+1} \in \mathbb{T}^i$  be the vector which consists of the first  $i$  coordinates of  $P_{i+1}$ . We also write  $B_p^i(A_{i+1}, r)$  for the  $\text{dist}_p$ -ball of radius  $r$  centered at  $A_{i+1}$  in the torus  $\mathbb{T}^i$ . Since  $\varepsilon < 1/2$ ,

$$\mathbb{E} \left| \frac{\partial f}{\partial x_{i+1}}(P_i) \right| = \mathbb{E} \frac{\int_{B_p^i(A_{i+1}, \frac{\varepsilon}{2^i}) \times \mathbb{T}^{n-i}} \left| \frac{\partial f}{\partial x_{i+1}} \right|}{\text{Vol}_i(B_p^i(A_{i+1}, \frac{\varepsilon}{2^i}))} = \frac{\mathbb{E} \int_{B_p^i(A_{i+1}, \frac{\varepsilon}{2^i}) \times \mathbb{T}^{n-i}} \left| \frac{\partial f}{\partial x_{i+1}} \right|}{\omega_{i,p} \left( \frac{\varepsilon}{2^i} \right)^i}. \quad (5)$$

From (4), (5) and the Hölder inequality, for  $i = 0, \dots, n-1$ ,

$$\begin{aligned} \mathbb{E}|f(P'_i) - f(P_i)| &\leq \left( \int_{\mathbb{T}^n} \left| \frac{\partial f}{\partial x_{i+1}} \right|^q \right)^{\frac{1}{q}} \left( \omega_{i,p} \left( \frac{\varepsilon}{2^i} \right)^i \right)^{-\frac{1}{q}} \\ &\leq \left( \frac{2^{i^2+qi}}{\omega_{i,p} \varepsilon^{i+q}} \right)^{-\frac{1}{q}} \left( \frac{\omega_{i,p} \varepsilon^i}{2^{i^2}} \right)^{-\frac{1}{q}} = \frac{\varepsilon}{2^i}, \end{aligned}$$

where we used our assumption (3) in the last passage. The function  $f$  is 1-Lipschitz with respect to  $\text{dist}_p$ , and hence  $|f(P'_i) - f(P_{i+1})| \leq \varepsilon/2^i$  with probability one. Consequently,

$$\mathbb{E}|f(P) - f(P_0)| \leq \mathbb{E} \sum_{i=0}^{n-1} |f(P_i) - f(P'_i)| + |f(P'_i) - f(P_{i+1})| \leq \sum_{i=0}^{n-1} \frac{2\varepsilon}{2^i} < 4\varepsilon.$$

However,  $P_0$  is distributed uniformly on the torus  $\mathbb{T}^n$ . Denote  $M = \mathbb{E}f(P_0) = \int_{\mathbb{T}^n} f$ . We have shown that  $|f(P) - M| < 4\varepsilon$ . Since  $P \in \mathbb{T}^n$  was an arbitrary fixed point, the lemma follows.  $\square$

For  $x \in \mathbb{T}^\infty$ , denote by  $F(x)$  the set of points in  $\mathbb{T}^\infty$  that coincide with  $x$  in all but finitely many coordinates.

**Lemma 4.** *Let  $A \subseteq \mathbb{T}^\infty$  satisfy  $\sigma(A) > 0$ . Then  $\sigma(\{x \in \mathbb{T}^\infty; F(x) \cap A \neq \emptyset\}) = 1$ .*

*Proof.* Denote  $B = \{x \in \mathbb{T}^\infty; F(x) \cap A \neq \emptyset\}$ . Then  $B$  is a measurable set which is in fact a tail event. Since  $A \subseteq B$ , Kolmogorov's zero-one law implies that  $\sigma(B) = 1$ .  $\square$

For a measure space  $X$  and a measurable function  $f : X \rightarrow \mathbb{R}$ , we define the essential supremum of  $f$ , denoted by  $\text{ess sup } f$ , as the supremum over all  $a \in \mathbb{R}$  for which the set  $\{x \in X; f(x) > a\}$  has a non-zero measure. The definition of essential infimum is analogous. Define the *essential oscillation* of  $f$  on  $X$  by

$$\text{essOsc}(f; X) = \text{ess sup } f - \text{ess inf } f.$$

Equivalently,  $\text{essOsc}(f; X) = \|f(x) - f(y)\|_{L^\infty(X \times X)}$ .

**Lemma 5.** *Let  $p \in (1, \infty]$ ,  $0 < \varepsilon < 1/2$  and let  $f : \mathbb{T}^\infty \rightarrow \mathbb{R}$  be 1-Lipschitz with respect to the metric  $\text{dist}_p$ . Denote  $q = p/(p-1)$ , with  $q = 1$  in case  $p = \infty$ . Assume that*

$$\int_{\mathbb{T}^\infty} \sum_{i=1}^{\infty} c_{\varepsilon,p,i} \cdot \left| \frac{\partial f}{\partial x_i} \right|^q \leq \frac{1}{2},$$

where  $c_{\varepsilon,p,i} = 2^{(i-1)^2+q(i-1)} / (\omega_{i-1,p} \cdot \varepsilon^{i-1+q})$ . Then

$$\text{essOsc}(f, \mathbb{T}^\infty) < 8\varepsilon. \quad (6)$$

*Proof.* Let  $a$  be a random point, distributed uniformly in  $\mathbb{T}^\mathbb{N}$ . For a subset  $S \subseteq \mathbb{N}$ , denote by  $a_S$  the restriction of the random point  $a$  to the torus  $\mathbb{T}^S$ . Define  $I_n = \{1, \dots, n\}$ . For  $b \in \mathbb{T}^{\mathbb{N} \setminus I_n}$  denote

$$\mathbb{T}^n \times \{b\} = \{x \in \mathbb{T}^\infty; x_i = b_i \forall i > n\}.$$

For  $n \geq 1$  we have

$$\begin{aligned} \mathbb{P} \left( \int_{\mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\}} \sum_{i=1}^n c_{\varepsilon,p,i} \left| \frac{\partial f}{\partial x_i} \right|^q \geq 1 \right) &\leq \mathbb{E}_a \int_{\mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\}} \sum_{i=1}^{\infty} c_{\varepsilon,p,i} \left| \frac{\partial f}{\partial x_i} \right|^q \\ &= \int_{\mathbb{T}^\mathbb{N}} \sum_{i=1}^{\infty} c_{\varepsilon,p,i} \left| \frac{\partial f}{\partial x_i} \right|^q \leq \frac{1}{2}. \end{aligned}$$

Lemma 3 now implies that for any  $n \geq 1$ ,

$$\mathbb{P} \left( \text{Osc} \left( f, \mathbb{T}^n \times \{a_{\mathbb{N} \setminus I_n}\} \right) < 8\varepsilon \right) \geq \frac{1}{2}.$$

Write  $B_n$  for the collection of all  $b \in \mathbb{T}^\infty$  for which  $\text{Osc}(f, \mathbb{T}^n \times \{b_{\mathbb{N} \setminus I_n}\}) < 8\varepsilon$ . Obviously  $B_{n+1} \subseteq B_n$ , and by the above  $\sigma(B_n) \geq \frac{1}{2}$  for all  $n \geq 1$ . Denoting  $B = \bigcap_{n=1}^{\infty} B_n$ , we have  $\sigma(B) \geq \frac{1}{2}$ . Note that

$$B = \{b \in \mathbb{T}^\mathbb{N}; |f(x) - f(y)| < 8\varepsilon \forall x, y \in F(b)\}. \quad (7)$$

If (6) does not hold, then there exist sets  $C, D \subseteq \mathbb{T}^{\mathbb{N}}$  of positive measure such that for all pairs of points  $c \in C$  and  $d \in D$  one has  $|f(c) - f(d)| \geq 8\varepsilon$ . Denote

$$\tilde{C} = \{x \in \mathbb{T}^{\mathbb{N}}; F(x) \cap C \neq \emptyset\} \quad \text{and} \quad \tilde{D} = \{x \in \mathbb{T}^{\mathbb{N}}; F(x) \cap D \neq \emptyset\}.$$

By Lemma 4, we have  $\sigma(\tilde{C}) = \sigma(\tilde{D}) = 1$ . Thus  $\sigma(B \cap \tilde{C} \cap \tilde{D}) \geq \frac{1}{2}$ , and there exist a point  $b \in B$  and two elements  $c \in C \cap F(b)$ ,  $d \in D \cap F(b)$ . According to the definition (7) of the set  $B$ ,

$$|f(c) - f(d)| < 8\varepsilon,$$

in contradiction.  $\square$

**Proposition 6.** *Let  $p \in (1, \infty]$ ,  $0 < \varepsilon < 1/2$  and let  $f : \mathbb{T}^{\infty} \rightarrow \mathbb{R}$  be a measurable function that is 1-Lipschitz with respect to  $\text{dist}_p$ . Then there exists a parallel infinite-dimensional subtorus  $M \subseteq \mathbb{T}^{\mathbb{N}}$  such that the restriction  $f|_M$  is measurable and  $\text{essOsc}(f; M) \leq 8\varepsilon$ .*

*Proof.* Fix a partition of  $\mathbb{N}$  into blocks  $B_1, B_2, \dots \subseteq \mathbb{N}$  of size

$$\#(B_n) = \left\lceil \frac{2^{(n-1)^2 + q(n-1) + (n+1)}}{\omega_{n-1} \cdot \varepsilon^{n-1+q}} \right\rceil \quad (n = 1, 2, \dots). \quad (8)$$

In each block, choose a random element  $i_n \in B_n$ , independently and uniformly. Denote  $I = \{i_1, i_2, \dots\} \subseteq \mathbb{N}$ . Additionally, let  $a$  be a random point, distributed uniformly in  $\mathbb{T}^{\mathbb{N}}$ , independent of  $I$ . As before, write  $q = p/(p-1)$  with  $q = 1$  in case  $p = \infty$ . For every fixed  $n$  and for every  $b \in \mathbb{T}^{\mathbb{N} \setminus B_n}$ , the function  $f$  restricted to  $\mathbb{T}^{B_n} \times \{b\}$  is 1-Lipschitz with respect to  $\text{dist}_p$ . By Rademacher's theorem, for almost any  $x \in \mathbb{T}^{B_n} \times \{b\}$  one has

$$\sum_{i \in B_n} \left| \frac{\partial f}{\partial x_i}(x) \right|^q \leq 1,$$

implying that

$$\mathbb{E}_{i_n} \int_{\mathbb{T}^{B_n} \times \{b\}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{\#(B_n)}. \quad (9)$$

Denote by  $a_{\mathbb{N} \setminus B_n}$  the restriction of the random point  $a$  to the torus  $\mathbb{T}^{\mathbb{N} \setminus B_n}$ . From (8) and (9),

$$\mathbb{E}_{i_n} \int_{\mathbb{T}^{\mathbb{N}}} \frac{2^{(n-1)^2 + q(n-1)}}{\omega_{n-1,p} \cdot \varepsilon^{n-1+q}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q = \mathbb{E}_{i_n, a} \int_{\mathbb{T}^{B_n} \times \{a|_{\mathbb{N} \setminus B_n}\}} \frac{2^{(n-1)^2 + q(n-1)}}{\omega_{n-1,p} \cdot \varepsilon^{n-1+q}} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2^{n+1}}.$$

Denote  $c_{\varepsilon,p,n} = 2^{(n-1)^2 + q(n-1)} / (\omega_{n-1,p} \cdot \varepsilon^{n-1+q})$ . Then,

$$\mathbb{E}_I \int_{\mathbb{T}^{\mathbb{N}}} \sum_{n=1}^{\infty} c_{\varepsilon,p,n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

That is,

$$\mathbb{E}_{I, a_{\mathbb{N} \setminus I}} \int_{\mathbb{T}^I \times \{a_{\mathbb{N} \setminus I}\}} \sum_{n=1}^{\infty} c_{\varepsilon,p,n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2}. \quad (10)$$

In particular, there exists a subset  $I = \{i_1, i_2, \dots\} \subseteq \mathbb{N}$  and  $b \in \mathbb{T}^{\mathbb{N}}$  such that

$$\int_{\mathbb{T}^I \times \{b_{\mathbb{N} \setminus I}\}} \sum_{n=1}^{\infty} c_{\varepsilon, p, n} \left| \frac{\partial f}{\partial x_{i_n}} \right|^q \leq \frac{1}{2}, \quad (11)$$

and such that the restriction of  $f$  to the subtorus  $M := \mathbb{T}^I \times \{b_{\mathbb{N} \setminus I}\}$  is measurable. We may apply Lemma 5 thanks to (11), and conclude that  $\text{essOsc}(f; M) < 8\varepsilon$ .  $\square$

*Proof of Theorem 2.* Normalizing, we may assume that  $f$  is 1-Lipschitz. Fix a sequence  $\varepsilon_n \rightarrow 0$  and apply Proposition 6 in order to construct a decreasing sequence of infinite-dimensional parallel tori  $T_n$  such that  $\text{essOsc}(f; T_n) < \varepsilon_n$ . Denote  $a_n = \int_{T_n} f$ . Then for  $m > n$  one has  $|a_m - a_n| < \varepsilon_n$ , implying that  $a_n$  has a limit, denoted by  $a$ . It then follows that  $a \in \mathbb{R}$  satisfies the conclusion of the theorem.  $\square$

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